Nonlinear Tracking Control of Underactuated Ships based on a Unified Kinematic and Dynamic Model

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Abstract—This paper proposes a nonlinear tracking control method for underactuated ships based on a unified kinematic and dynamic model. There are several literatures using backstepping-based controllers, which usually results in the complicated control structures and the restriction on the reference trajectories. To obtain a controller with much simpler structure and less restriction on the reference trajectory, the kinematics and dynamics are combined into a single unified model and then backstepping-like linearization method is employed based on this model. Whereas the torque input for translation motion can be directly obtained, the torque input for rotational motion is designed in such a way that the pseudo yaw angle command is first designed and then the actual yaw angle is made to follow its pseudo command. The resulting position and yaw angle tracking errors are shown to remain globally uniformly ultimately bounded through stability analysis. Furthermore, the choice of the control gains is less restrictive than the previous similar methods, and more general reference trajectory can be followed by the proposed controller, as demonstrated in the numerical simulations.

I. INTRODUCTION

Systems with fewer number of independent actuators than degrees of freedom are known as underactuated mechanical systems, which includes two-wheeled mobile robots with nonholonomic velocity constraints [1-3] and underactuated ships with nonholonomic velocity and acceleration constraints [4-13]. In particular, underactuated ship is more difficult to apply the control methods than the wheeled mobile robot, in the sense that the underactuated ship cannot be transformed into a driftless chained system [4] and the class of employable control approaches becomes smaller in the case of the underactuated system. Furthermore, as it cannot be asymptotically stabilized by a continuous time-invariant feedback law [14], control of underactuated surface ships with only surge force and yaw moment have been much studied.

The results in [6-10] studied the global tracking problem, but showed the limited performance in tracking a straight-line reference trajectory since they required the reference trajectory to be persistently exciting. Whereas the stabilization, tracking, or path-following controllers in [11-13] are either too complicated for implementation or too dependent on the reference trajectory.

In this paper, a backstepping-like linearization method is proposed based on the unified kinematic and dynamic model. As will be shown later in Fig. 1 in Section III, the ship model obtained by combining the kinematics and dynamics is used to obtain the torque input for translational motion and the pseudo yaw angle command. Then, the torque input for rotational motion is designed for the tracking of actual yaw angle to the pseudo yaw angle command. The stability analysis and simulation results show that the tracking performance for reference trajectories such as circles, straight lines, and a combinations of these trajectories can be improved, while the control structure is simpler and general reference trajectory can be followed.

II. KINEMATICS AND DYNAMICS OF UNDERACTUATED SHIPS

This section briefly describes the kinematics and dynamics of underactuated ships and then presents a unified kinematic and dynamic model for controller design. The following assumptions are made to reduce the general six degree-of-freedom ship model into the motion in surge, sway, and yaw: i) the heave, roll, and pitch modes induced by wave, wind, and currents drift forces can be neglected; ii) the inertia, added mass, and hydrodynamic damping matrices are diagonal, which holds good for ships having port/starboard and fore/aft symmetry; and iii) available controls are surge force $\tau_u$ and yaw moment $\tau_r$. Then, as in [15], the kinematic and dynamic models can be given by

$$\begin{align*}
\dot{x} &= u \cos(\psi) - v \sin(\psi) \\
y &= u \sin(\psi) + v \cos(\psi) \\
\dot{\psi} &= r
\end{align*}$$

(1)

$$\begin{align*}
\dot{u} &= \frac{m_2}{m_1} v r - \frac{d_3}{m_1} u + \frac{1}{m_1} \tau_u \\
\dot{v} &= -\frac{m_1}{m_2} u r - \frac{d_2}{m_2} v \\
\dot{\tau} &= \frac{(m_1 - m_2)}{m_3} u v - \frac{d_1}{m_3} r + \frac{1}{m_3} \tau_r
\end{align*}$$

(2)

Here, $x$ and $y$ are surge and sway displacements, respectively, and $\psi$ is yaw angle of the ship in the earth-fixed frame; $u$ and $v$ are linear velocities in surge (body-fixed $x$ -direction) and sway (body-fixed $y$ -direction),
and $r$ is angular velocity in yaw (body-fixed $z$-direction); and the parameters $m_i$ are given by inertia and added mass effects, and $d_i$ are given by hydrodynamic damping effects satisfying $m_i, d_i > 0$ for $i = 1, 2, 3$. Unlike the wheeled mobile robot, the time derivatives of surge and sway displacement in (1) contain the sway velocity components and the second equation in (2) shows that this sway velocity decreases to zero only when the surge velocity $u$ or the angular velocity $r$ remains zero. Thus, the control law for the mobile robot cannot be directly employed to the ship model, and the sway motion must be considered in the design of the ship control law.

III. TRACKING CONTROL FOR A UNIFIED KINEMATIC AND DYNAMIC MODEL

We consider the following continuously differentiable reference trajectory $(x_d, y_d, \psi_d)$, which is assumed to be feasible by a proper choice of reference surge force $u_{rd}$ and yaw moment $r_{rd}$.

\[
\begin{align*}
\dot{x}_d &= u_d \cos(\psi_d) - v_d \sin(\psi_d) \\
\dot{y}_d &= u_d \sin(\psi_d) + v_d \cos(\psi_d) \\
\dot{\psi}_d &= r_d \\
\end{align*}
\]

(3)

where $x_d$ and $y_d$ are reference surge and sway displacements, $\psi_d$ is a reference yaw angle, $u_d$ and $v_d$ are reference surge and sway linear velocities, and $r_d$ is the reference angular velocity.

The control objective is to make the ship track the reference trajectory, which implies that the position tracking errors

\[
X := x - x_d, \quad Y := y - y_d
\]

(5)

converge to zero while the yaw angle is maintained at the desired direction. In the proposed method as shown in Fig. 1, $\tau_u$ and $\psi_{pseudo}$ are obtained in the kinematic linearization, and then, $\tau_u$ is obtained in terms of $\psi_{pseudo}$ in the dynamic linearization.

Combining (1) and (2), we have

\[
\begin{align*}
\dot{x} &= \left( m_1 v - \frac{d_1}{m_1} u + \frac{1}{m_1} \tau_u \right) \cos \psi - ur \sin \psi \\
\dot{y} &= \left( m_1 v - \frac{d_1}{m_1} u + \frac{1}{m_1} \tau_u \right) \sin \psi + ur \cos \psi \\
\dot{\psi} &= \left( m_1 v - \frac{d_1}{m_1} u + \frac{1}{m_1} \tau_u \right) \cos \psi - vr \sin \psi \\
\end{align*}
\]

(6)

If we introduce the variables

\[
\begin{align*}
X &= \hat{x}_d - ax_d - bx_d + \left( \frac{d_1}{m_1} u \right) \cos \psi + ur \sin \psi \\
&\quad + \left( -\frac{m_1 u}{m_2} - \frac{d_2}{m_2} v \right) \sin \psi + vr \cos \psi \\
Y &= \hat{y}_d - ay_d - by_d + \left( \frac{d_1}{m_1} u \right) \sin \psi - ur \cos \psi \\
&\quad - \left( -\frac{m_1 u}{m_2} - \frac{d_2}{m_2} v \right) \cos \psi + vr \sin \psi
\end{align*}
\]

(7)

then the first and second rows of (6) become

\[
\begin{align*}
\dot{x}_r &= a \hat{x}_r + b x_r \\
\dot{y}_r &= a \hat{y}_r + b y_r
\end{align*}
\]

(8)

where $\tau_u := (m_1 v + r_u/m_1).

Since we want $\dot{x}$ and $\dot{y}$ to be equal to $\hat{x}_d - ax_d - bx_d$, $\hat{y}_d - ay_d - by_d$ for $a, b > 0$ to have $\dot{x}_r + a \hat{x}_r + bx_r = 0$ and $\dot{y}_r + a \hat{y}_r + by_r = 0$ for position tracking, the control input $r_u$ (i.e., $\tau_u$) and $\psi$ should satisfy

\[
\cos(\psi) \hat{\tau}_u = X \\
\sin(\psi) \hat{\tau}_u = Y
\]

(9)

Using (9), $\hat{\tau}_u$ can be selected as

\[
\hat{\tau}_u = \cos(\psi) X + \sin(\psi) Y
\]

(10)

Thus, $\tau_u$ can be obtained as

\[
\tau_u = m_1 [\cos(\psi) X + \sin(\psi) Y] - m_1 vr
\]

(11)

Next, $\tau_u$ needs to be designed such that $\psi$ follows the pseudo heading direction angle $\psi_{pseudo}$. To this end, considering both position tracking and yaw angle tracking, $\psi_{pseudo}$ can be determined as follows. Using $\psi^* = \text{atan2}(Y, X)$ where $\text{atan2}(y, x)$ is a four-quadrant inverse tangent with the values in the interval of $(-\pi, \pi)$, (10) becomes

\[
\hat{\tau}_u = \cos(\psi - \psi^*) \sqrt{X^2 + Y^2}
\]

(12)

which is positive for $|\psi - \psi^*| < \frac{\pi}{2}$ and negative for $|\pi - (\psi - \psi^*)| < \frac{\pi}{2}$. Here, it should be noted that the first row of (2) can be changed as $u = -\frac{d_1}{m_1} u + \tau_u$, which implies that $u$ has a low-pass filtered value of $\hat{\tau}_u$ such that the sign of $\hat{\tau}_u$ determines the sign of $u$. Thus, when
\( u_d \) is positive (negative), \( u \) should be positive (negative), i.e., \( \psi - \psi^* \) should satisfy \( |\psi - \psi^*| < \frac{\pi}{2} \) (12). Also, from (9), we have
\[
\sin(\psi)X - \cos(\psi)Y = \sin(\psi - \psi^*)\sqrt{X^2 + Y^2} = 0 \ ,
\]
which implies that \( \psi \) should be \( \psi^* \) or \( \psi^* + \pi \). Thus, \( \psi_{\text{pseudo}} \) should be chosen as
\[
\psi_{\text{pseudo}} = \psi^* \quad \text{(13a)}
\]
in the case of positive \( u_d \), and
\[
\psi_{\text{pseudo}} = \psi^* + \pi \quad \text{(13b)}
\]
in the case of negative \( u_d \), such that \( \sin(\psi_{\text{pseudo}})X - \cos(\psi_{\text{pseudo}})Y = 0 \) is satisfied. It should be noted here that larger class of reference trajectories can be tracked, in the sense that the assumption on the yaw angle tracking error (\( \psi_e := \psi - \psi^* \)) as \( |\psi_e| < 0.5 \) in (16) are not needed here. It should be noted that the information of \( \dot{x}_e \) and \( \dot{y}_e \) in (7) (or \( \dot{x} \) and \( \dot{y} \)) can be obtained from \( u \) and \( v \) using the first and second rows in (1).

In the following, the control input \( \tau_e \) is designed to make \( \psi \) follow \( \psi_{\text{pseudo}} \). Instead of backstepping or recursive design, we propose a high-gain design feedback to dominate \( \psi_{\text{pseudo}} \) as in [17]. Whereas \( x \) - and \( y \) -dynamics can be controlled by \( \tau_x \) in (10), \( \psi \) -dynamics can be controlled by \( \tau \), given by
\[
\tau = m_3 \left\{ \frac{(m_1 - m_2)}{m_3} \nu v + \frac{d_1}{m_3} r - ar - b \sin E_y \right\}
\]
(14)
such that
\[
\dot{\psi} = r
\]
\[
\dot{r} = -ar - b \sin E_y
\]
(15)
where \( E_y := \psi - \psi_{\text{pseudo}} \), \( a := 0.9k_r \), and \( b := k^2 \) are positive constants.

Now, we can have the stability and performance of the closed-loop control system as in the following theorem.

**Theorem 1 (Tracking Control):** If the control inputs \( \tau = [\tau_x, \tau_y]^T \) are used where \( \tau_x \) in (11), \( \tau_y \) in (14), and \( \psi_{\text{pseudo}} \), in (13), then the global ultimate boundedness of the position tracking errors (5) and yaw angle tracking error can be achieved. The ultimate bounds of the tracking errors are dependent on \( E_y \), \( \dot{x}_d, \dot{y}_d \), and can be made smaller with the proper choice of \( k_r \) (or \( a \) and \( b \)).

The magnitude of \( E_y \) is dependent on \( b \psi_{\text{pseudo}} + a \psi_{\text{pseudo}} + \psi_{\text{pseudo}}^* \), where \( \psi_{\text{pseudo}} \) and \( \psi_{\text{pseudo}}^* \) are first- and second-order time derivatives of \( \psi_{\text{pseudo}} \). In particular, when the reference trajectory is straight line with constant velocity (i.e., \( \dot{x}_d = \dot{y}_d = 0 \)), \( b \psi_{\text{pseudo}} + a \psi_{\text{pseudo}} + \psi_{\text{pseudo}}^* \) reduces to zero as the position tracking is achieved, and accordingly, the tracking errors reduce to zero.

**Proof:** (8) and (10) give (16). Here, the fourth equality follows from \( \psi_{\text{pseudo}} = \text{atan}(Y, X) \), i.e.,
\[
\sin(\psi_{\text{pseudo}})X = \cos(\psi_{\text{pseudo}})Y \quad \text{Eqn. (16) can be expressed as}
\]
\[
\hat{\zeta}_i = A\zeta_i + B_1[\cos(\psi_{\text{pseudo}})]X + \sin(\psi_{\text{pseudo}})Y \sin(E_y/2)
\]
(17)
where \( i = 1, 2 \), \( \zeta_i = [x_i, \dot{x}_i]^T \), \( \zeta_2 = [y_i, \dot{y}_i]^T \),
\[
A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -2\sin(\psi) \end{bmatrix} \quad B_2 = \begin{bmatrix} 2\cos(\psi) \end{bmatrix}.
\]
Now, we can choose the Lyapunov candidate as
\[
V = \frac{1}{2}(\zeta_i^T P\zeta_i + \zeta_2^T P\zeta_2) \quad \text{where } P > 0 \text{ satisfies}
\]
\[
A^T P + PA = -Q > 0 \quad \text{since } A \text{ is stable. Then, we have}
\]
\[
\dot{V} = -\zeta_i^T Q\zeta_i - \zeta_2^T Q\zeta_2 + 2(\zeta_i^T PB_1 + \zeta_2^T PB_2)[\cos(\psi_{\text{pseudo}})]X + \sin(\psi_{\text{pseudo}})Y \sin(E_y/2)
\]
(18)
\[
+ \sin((\psi_{\text{pseudo}})/2)]Y \sin(E_y/2).
\]
It should be noted here that as \( r \) remains bounded from the second row of (15), and \( \dot{x}_d, \dot{y}_d \) are bounded, \( X \) and \( y \) are bounded.
Y in (7) are bounded with respect to $\zeta_1$ and $\zeta_2$. Thus, if $E_{\psi}$ is sufficiently reduced by the control input for rotational motion, then it can be guaranteed that $\zeta_1$ can be ultimately bounded with respect to $\dot{x}_d, \dot{y}_d, r$, which can be more easily seen when $Q$ is chosen to be a diagonal matrix. The ultimate bound is dependent on $E_{\psi}, \dot{x}_d, \dot{y}_d$ and can be adjusted by the proper choice of $k_r$ (note that $a, b$ used in (15) are functions of $k_r$). Accordingly, it is desirable to reduce $E_{\psi}$ to reduce the ultimate bound.

Finally, we choose the Lyapunov candidate as $W = (a^2/2 + b)\xi_1^2 + c(1 - \cos(\xi_1))^2 - (b\psi_{\text{pseudo}} + a\psi'_{\text{pseudo}} + \psi_{\text{pseudo}}^2)$ from (16). We can suppose here that $-\pi < \xi_1 \leq \pi$ without loss of generality, since another Lyapunov candidate can be selected in the other interval of $\xi_1$. Then, the time derivative of $W$ becomes

$$W = (a^2/2 + b)\dot{\xi}_1 \xi_1 + (a\xi_1 \xi_2 + a\xi_2 \xi_1 + 2c\sin(\xi_1)\dot{\xi}_1)
= (a^2/2 + b)\xi_1^2 + c(1 - \cos(\xi_1))^2 - (b\psi_{\text{pseudo}} + a\psi'_{\text{pseudo}} + \psi_{\text{pseudo}}^2)
= -ac\psi_{\text{pseudo}}\sin(\xi_1) - ab\xi_1^2 - a\xi_2^2 - (a\xi_1 + 2\xi_2)\cdot(b\psi_{\text{pseudo}} + a\psi'_{\text{pseudo}} + \psi_{\text{pseudo}}^2).$$

As $\xi_1 \sin(\xi_1/2) \geq 0$ holds from $-\pi < \xi_1 \leq \pi$, the ultimate bound of $\xi_1$ becomes dependent on $b\psi_{\text{pseudo}} + a\psi'_{\text{pseudo}} + \psi_{\text{pseudo}}^2$, which reduces to zero when tracking the straight line, which can be also seen in the simulation results in the next section. Thus, it can be guaranteed that tracking errors converge to zero when the reference trajectory is a straight line with constant velocity.

It should be noted that the global ultimate boundedness of tracking errors were obtained by simplifying the control structure, compared with the previous backstepping/cascade methods [11-13].

IV. SIMULATION RESULTS

In this section, we present the simulation results of the proposed method for the underactuated ship. The parameters of the kinematics (1) and the dynamics (2) are the same as those in [13]: $m_1 = 120 \times 10^4 \text{kg}$, $m_2 = 172.9 \times 10^4 \text{kg}$, $m_3 = 636 \times 10^4 \text{kg} \cdot \text{m}^2$, $d_1 = 215 \times 10^7 \text{kg} \cdot \text{s}^{-1}$, $d_2 = 97 \times 10^4 \text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$, $d_3 = 802 \times 10^4 \text{m}^2 \cdot \text{s}^{-1}$. The controller is given by (11), (14), (7a), and (7b), and the design parameter is chosen as $k_r = 5$. The reference trajectory is generated using the reference model in (3), where the desired sway velocity is fixed as $v_d = 0$ to generate more general trajectories. Initial position and yaw angle tracking errors are assumed to exist in several scenarios with initial velocities set to be $(u(0), v(0), r(0)) = (0, 0, 0)$, and the disturbances given as in [13] are included, details of which are omitted. Here, we assume that the actuator dynamics are ideal and state variables are available. When the state variables are not available, state estimator may have to be designed for practical implementation. It is also noted that practical conditions such as (kinematic and dynamic) uncertainties and noise needs to be considered by involving adaptive and/or robust control methods. These issues can be studied in the future work. Here, the reference trajectories of each scenario are i) sinusoidal curve, ii) a straight line combined with the circle, and iii) a straight line with backward motion.

Fig. 2 shows the results for Scenario 1, where a sinusoidal reference trajectory with the time varying velocity is used. The position tracking errors remain sufficiently near zero, and the yaw angle tracking error remains at around 0 degree in Figs. 2 (a) and (b). As stated in Theorem 1, $\psi_{\text{pseudo}} + a\psi'_{\text{pseudo}} + \psi_{\text{pseudo}}^2$ determines the ultimate bounds of the position and yaw angle tracking errors. Thus, as the curvature of the reference trajectory increases, $\psi_{\text{pseudo}}$ (or the reference yaw velocity) becomes time-varying and thus the yaw angle tracking error does not converge to zero. Even in this case, position tracking errors remain sufficiently small and the surge direction is maintained in the desired forward direction by generating $\psi_{\text{pseudo}}$ as (13).

In Fig. 3, the position and yaw angle tracking errors are sufficiently small for tracking the straight line combined with the circle. In addition to the advantage of much simpler control structure, the tracking performance is relatively faster, when compared with the controller in [11]. Fig. 3 (d) shows that $\psi_e$ converges to zero and then it remains bounded, as $\psi_{\text{pseudo}}$ becomes first constant in tracking the straight line and then time-varying in tracking the circle. From the viewpoint of path-following, the performance in Fig. 3 (a) is quite satisfactory.

Fig. 4 shows the tracking performance of the straight line with backward motion where $\psi_{\text{pseudo}}$ is selected as (13b). Note that the straight line with forward motion is satisfactory in Fig. 3 (a) for Scenario 2. Fig. 4 (d) shows that $\psi_e$ is maintained sufficiently at near zero. Unlike the results in [16] which requires the yaw angle tracking error...
to be $|\psi_\theta| < 0.5\pi$, $\psi_e(0) > 0.5\pi$ is used in Scenario 3. This shows that our scheme can be applied for a whole range of the yaw angle tracking error by determining $\psi_{\text{pseudo}}$ as (13).

V. CONCLUSIONS

In this paper, we proposed a nonlinear tracking control method for underactuated ships based on a unified kinematic and dynamic model. Global ultimate uniform boundedness of the tracking errors in both position and yaw angle is obtained through Lyapunov analysis. By combining the kinematics and dynamics as a single model, we could employ the simplified backstepping-like linearization method such that the torque input for forward translational motion can be directly obtained. In this way, unlike the previous results, both reference trajectories and the design parameters are not restricted in the proposed method, which is also demonstrated in the numerical simulations. In further studies, the compensation for the uncertainties in kinematics and dynamics input constraints needs to be pursued, and the state observation problem is also an important issue.

ACKNOWLEDGMENT

This work was supported by grant No. R01-2006-000-11373-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

REFERENCES


Fig. 2. Performance of the proposed controller (Scenario 1).
Fig. 3. Performance of the proposed controller (Scenario 2).

Fig. 4. Performance of the proposed controller (Scenario 3).